

# $\rho$ -assoc and $\rho$ -dist of $wfs$ and $f$ in $\Sigma$ and $\mathcal{L}_{\mathcal{H}\mathcal{A}}$ -theory on 0-OL

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## Abstract

In this paper we create pseudo associativity ( $\rho$ -assoc) and pseudo distributivity ( $\rho$ -dist) properties for not fundamental operators NFO  $\downarrow, \uparrow$ , using two semantic rules, also we build the proofs for this result in Hilbert-Ackermann ( $\mathcal{H}\mathcal{A}$ ) axiomatic system, all this in the 0-order logic (0-OL) context.

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## 1 Introduction

In 0-OL exists classic results about basic properties of associativity and distributivity with  $\vee, \wedge, \rightarrow, \leftrightarrow$  and  $\neg$  operators [2] these are a consequence of the semantic (truth tables [1]) and syntactic ( $\mathcal{H}\mathcal{A}$  axiomatic system), in this paper we show a new notion about the associative and distributive properties for not fundamental operators (NFO).

**Definition 1.1** (NFO, FO). *Are binary operators*

*NFO are the operators  $\downarrow, \uparrow, \leftarrow, \oplus$  the negations forms of the FO*

*FO are the classic operators  $\vee, \wedge, \rightarrow, \leftrightarrow$*

**Definition 1.2** ( $\Sigma_{NFO}$ ,  $\Sigma_{FO}$ ). *Are languages [2]*

*$\Sigma_{NFO}$  is the language with NFO, monary operator  $\neg$  and parentheses.*

*$\Sigma_{FO}$  is the language with FO, monary operator  $\neg$  and parentheses.*

The NFO and FO have dual representations in a  $\Sigma$  language.

## 2 Semantic comparison

**Definition 2.1** ( $\mathcal{A}'$ -wfs of  $\Sigma_{NFO}$ ). *A  $\mathcal{A}'$ -wfs is a recursive string of the 0-OL semantic balanced and structurally well formed with interpretation [1] that has the following elements.*

1. Atoms:  $p, q, \dots$  that represent statements
2. Symbols of  $\Sigma_{NFO}$

**Definition 2.2** ( $\mathcal{A}$ -wfs of  $\Sigma_{FO}$ ). A  $\mathcal{A}$ -wfs is a recursive string of the 0-OL semantic balanced and structurally well formed with interpretation that has the following elements.

1. Atoms:  $p, q, \dots$  that represent statements
2. Symbols of  $\Sigma_{FO}$

*Note.* The  $\neg$  operator changes the interpretation of 1 to 0 and viceversa.

**Definition 2.3** (Truth Table [1]). Graphical format for strings  $\mathcal{A}$  or  $\mathcal{A}'$ , containing all possible values of interpretations of the atoms  $\mathcal{I}(p, q, \dots)$  and the interpretations of operators. The following are the truth tables for NFO of  $\mathcal{A}'$ -wfs and FO of  $\mathcal{A}$ -wfs. The final analysis is represented by the darker color column.

$p$	$\downarrow$	$q$	$p$	$\uparrow$	$q$	$p$	$\leftarrow$	$q$	$p$	$\oplus$	$q$
1	0	1	1	0	1	1	0	1	1	0	1
1	0	0	1	1	0	1	1	0	1	1	0
0	0	1	0	1	1	0	0	1	0	1	1
0	1	0	0	1	0	0	0	0	0	0	0

  

$p$	$\vee$	$q$	$p$	$\wedge$	$q$	$p$	$\rightarrow$	$q$	$p$	$\leftrightarrow$	$q$
1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	0	0	1	0	0	1	0	0
0	1	1	0	0	1	0	1	1	0	0	1
0	0	0	0	0	0	0	1	0	0	1	0

To simplify writing let  $\mathcal{E}$  a primitive symbol that describes “are wfs of”

**Definition 2.4** (Semantic Parallel).  $\mathcal{A}' \mathcal{E} \Sigma_{NFO}$  is the parallel of  $\mathcal{A} \mathcal{E} \Sigma_{FO}$  iff  $\mathcal{I}(\mathcal{A}')$  is equal to  $\mathcal{I}(\mathcal{A})$  for all values of the atoms in the final analysis of  $\mathcal{A}$  and  $\mathcal{A}'$ , the parallel is denoted by  $\mathcal{A} \parallel \mathcal{A}'$ .

**Definition 2.5** (Semantic Perpendicularity).  $\mathcal{A}' \mathcal{E} \Sigma_{NFO}$  is the perpendicular of  $\mathcal{A} \mathcal{E} \Sigma_{FO}$  iff  $\mathcal{I}(\mathcal{A}')$  is equal to  $\mathcal{I}(\neg \mathcal{A})$  for all values of the atoms in the final analysis of  $\mathcal{A}$  and  $\mathcal{A}'$ , the perpendicularity is denoted by  $\mathcal{A} \perp \mathcal{A}'$ .

**Definition 2.6** (Tautology).  $\mathcal{A}$ -wfs or  $\mathcal{A}'$ -wfs are tautology if the interpretation  $\mathcal{I}(\mathcal{A}) = 1$  or  $\mathcal{I}(\mathcal{A}') = 1$  respectively for all values of the final analysis.

**Definition 2.7** (Contradiction).  $\mathcal{A}$ -wfs or  $\mathcal{A}'$ -wfs is a contradiction if the interpretation  $\mathcal{I}(\mathcal{A}) = 0$  or  $\mathcal{I}(\mathcal{A}') = 0$  respectively for all values of the final analysis.

**Proposition 2.8.** Associativity and distributive properties are tautologies [4] [7] in the 0-OL semantic with FO i.e.

$$\mathcal{A}_1 \mathcal{I}((p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)) = 1$$

$$\mathcal{A}_2 \quad \mathcal{J}((p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)) = 1$$

$$\mathcal{A}_3 \quad \mathcal{J}((p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))) = 1$$

$$\mathcal{A}_4 \quad \mathcal{J}((p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))) = 1$$

*Proof.* With truth tables can be verified

	$(p$	$\vee$	$(q$	$\vee$	$r))$	$\leftrightarrow$	$((p$	$\vee$	$q)$	$\vee$	$r)$
	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	0	1	1	1	1	1	0
	1	1	0	1	1	1	1	1	0	1	1
$\mathcal{A}_1$	1	1	0	0	0	1	1	1	0	1	0
	0	1	1	1	1	1	0	1	1	1	1
	0	1	1	1	0	1	0	1	1	1	0
	0	1	0	1	1	1	0	0	0	1	1
	0	0	0	0	0	1	0	0	0	0	0

	$(p$	$\wedge$	$(q$	$\wedge$	$r))$	$\leftrightarrow$	$((p$	$\wedge$	$q)$	$\wedge$	$r)$
	1	1	1	1	1	1	1	1	1	1	1
	1	0	1	0	0	1	1	1	1	0	0
	1	0	0	0	1	1	1	0	0	0	1
$\mathcal{A}_2$	1	0	0	0	0	1	1	0	0	0	0
	0	0	1	1	1	1	0	0	1	0	1
	0	0	1	0	0	1	0	0	1	0	0
	0	0	0	0	1	1	0	0	0	0	1
	0	0	0	0	0	1	0	0	0	0	0

	$(p$	$\vee$	$(q$	$\wedge$	$r))$	$\leftrightarrow$	$((p$	$\vee$	$q)$	$\wedge$	$(p$	$\vee$	$r))$
$\mathcal{A}_3$	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	1	1	1	1	1	1	1	0
	1	1	0	0	1	1	1	1	0	1	1	1	1
	1	1	0	0	0	1	1	1	0	1	1	1	0
	0	1	1	1	1	1	0	1	1	1	0	1	1
	0	0	1	0	0	1	0	1	1	0	0	0	0
	0	0	0	0	1	1	0	0	0	0	0	1	1
	0	0	0	0	0	1	0	0	0	0	0	0	0

	$(p$	$\wedge$	$(q$	$\vee$	$r))$	$\leftrightarrow$	$((p$	$\wedge$	$q)$	$\vee$	$(p$	$\wedge$	$r))$
$\mathcal{A}_4$	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	0	1	1	1	1	1	1	0	0
	1	1	0	1	1	1	1	0	0	1	1	1	1
	1	0	0	0	0	1	1	0	0	0	1	0	0
	0	0	1	1	1	1	0	0	1	0	0	0	1
	0	0	1	1	0	1	0	0	1	0	0	0	0
	0	0	0	1	1	1	0	0	0	0	0	0	1
	0	0	0	0	0	1	0	0	0	0	0	0	0

■

**Proposition 2.9.** *We can create a  $\rho$ -assoc and  $\rho$ -dist properties in the  $\theta$ -OL semantic with NFO that are contradictions*

$$\mathcal{A}'_1 \quad \mathcal{J}((p \downarrow \neg(q \downarrow r)) \oplus (\neg(p \downarrow q) \downarrow r)) = 0$$

$$\mathcal{A}'_2 \quad \mathcal{J}((p \uparrow \neg(q \uparrow r)) \oplus (\neg(p \uparrow q) \uparrow r)) = 0$$

$$\mathcal{A}'_3 \mathcal{I}((p \downarrow \neg(q \uparrow r)) \oplus (\neg(p \downarrow q) \uparrow \neg(p \downarrow r))) = 0$$

$$\mathcal{A}'_4 \mathcal{I}((p \uparrow \neg(q \downarrow r)) \oplus (\neg(p \uparrow q) \downarrow \neg(p \uparrow r))) = 0$$

*Proof.* With truth tables can be verified

$\mathcal{A}'_1$	$(p$	$\downarrow$	$\neg(q$	$\downarrow$	$r))$	$\oplus$	$(\neg(p$	$\downarrow$	$q)$	$\downarrow$	$r)$
	1	0	1	1	1	0	1	1	1	0	1
	1	0	1	1	0	0	1	1	1	0	0
	1	0	0	1	1	0	1	1	0	0	1
	1	0	0	0	0	0	1	1	0	0	0
	0	0	1	1	1	0	0	1	1	0	1
	0	0	1	1	0	0	0	1	1	0	0
	0	0	0	1	1	0	0	0	0	0	1
	0	1	0	0	0	0	0	0	0	1	0

$\mathcal{A}'_2$	$(p$	$\uparrow$	$\neg(q$	$\uparrow$	$r))$	$\oplus$	$(\neg(p$	$\uparrow$	$q)$	$\uparrow$	$r)$
	1	0	1	1	1	0	1	1	1	0	1
	1	1	1	0	0	0	1	1	1	1	0
	1	1	0	0	1	0	1	0	0	1	1
	1	1	0	0	0	0	1	0	0	1	0
	0	1	1	1	1	0	0	0	1	1	1
	0	1	1	0	0	0	0	0	1	1	0
	0	1	0	0	1	0	0	0	0	1	1
	0	1	0	0	0	0	0	0	0	1	0

$\mathcal{A}'_3$	$(p$	$\downarrow$	$\neg(q$	$\uparrow$	$r))$	$\oplus$	$(\neg(p$	$\downarrow$	$q)$	$\uparrow$	$\neg(p$	$\downarrow$	$r))$
	1	0	1	1	1	0	1	1	1	0	1	1	1
	1	0	1	0	0	0	1	1	1	0	1	1	0
	1	0	0	0	1	0	1	1	0	0	1	1	1
	1	0	0	0	0	0	1	1	0	0	1	1	0
	0	0	1	1	1	0	0	1	1	0	0	1	1
	0	1	1	0	0	0	0	1	1	1	0	0	0
	0	1	0	0	1	0	0	0	0	1	0	1	1
	0	1	0	0	0	0	0	0	0	1	0	0	0

$\mathcal{A}'_4$	$(p$	$\uparrow$	$\neg(q$	$\downarrow$	$r))$	$\oplus$	$(\neg(p$	$\uparrow$	$q)$	$\downarrow$	$\neg(p$	$\uparrow$	$r))$
	1	0	1	1	1	0	1	1	1	0	1	1	1
	1	0	1	1	0	0	1	1	1	0	1	0	0
	1	0	0	1	1	0	1	0	0	0	1	1	1
	1	1	0	0	0	0	1	0	0	1	1	0	0
	0	1	1	1	1	0	0	0	1	1	0	0	1
	0	1	1	1	0	0	0	0	1	1	0	0	0
	0	1	0	1	1	0	0	0	0	1	0	0	1
	0	1	0	0	0	0	0	0	0	1	0	0	0

■

**Theorem 2.10.**  $\mathcal{A}_1 \perp \mathcal{A}'_1, \mathcal{A}_2 \perp \mathcal{A}'_2, \mathcal{A}_3 \perp \mathcal{A}'_3, \mathcal{A}_4 \perp \mathcal{A}'_4$

*Proof.* Clearly  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4 \in \Sigma_{FO}$  and  $\mathcal{A}'_1, \mathcal{A}'_2, \mathcal{A}'_3, \mathcal{A}'_4 \in \Sigma_{NFO}$  also

$$\mathcal{I}(\mathcal{A}_1) = \mathcal{I}(\neg \mathcal{A}'_1) = 1$$

$$\mathcal{I}(\mathcal{A}_2) = \mathcal{I}(\neg \mathcal{A}'_2) = 1$$

$\mathcal{I}(\mathcal{A}_3) = \mathcal{I}(\neg \mathcal{A}'_3) = 1$   
 $\mathcal{I}(\mathcal{A}_4) = \mathcal{I}(\neg \mathcal{A}'_4) = 1$   
 Then by definition of  $\perp$   
 $\mathcal{A}_1 \perp \mathcal{A}'_1, \mathcal{A}_2 \perp \mathcal{A}'_2, \mathcal{A}_3 \perp \mathcal{A}'_3, \mathcal{A}_4 \perp \mathcal{A}'_4$  ■

**Corollary 2.11.**  $\mathcal{A}_1 \parallel \neg \mathcal{A}'_1, \mathcal{A}_2 \parallel \neg \mathcal{A}'_2, \mathcal{A}_3 \parallel \neg \mathcal{A}'_3, \mathcal{A}_4 \parallel \neg \mathcal{A}'_4$

**Corollary 2.12.** Let  $\mathcal{A} \in \Sigma_{FO}$  and  $\mathcal{A}' \in \Sigma_{NFO}$

- a)  $\mathcal{A} \perp \mathcal{A}'$  iff  $\mathcal{A} \parallel \neg \mathcal{A}'$
- b)  $\mathcal{A} \parallel \neg \mathcal{A}'$  iff  $\neg \mathcal{A} \parallel \mathcal{A}'$

We proceed to create two semantic rules of replacement that guarantee a  $\rho$ -assoc and  $\rho$ -dist with NFO.

**Definition 2.13.** Let  $\Upsilon$  the property that encrypts  $\neg$  monary operator in wfs  $\mathcal{A}' \in \Sigma_{NFO}$  iff this precedes a parentheses with two  $\mathcal{A}'$ -wfs which can be atoms operated by  $\downarrow$  or  $\uparrow$  NFOs, but does not change the interpretation.

- $\Upsilon((p \downarrow \neg(q \downarrow r)) \oplus (\neg(p \downarrow q) \downarrow r))$  is  $(p \downarrow (q \downarrow r)) \oplus ((p \downarrow q) \downarrow r)$
- $\Upsilon((p \uparrow \neg(q \uparrow r)) \oplus (\neg(p \uparrow q) \uparrow r))$  is  $(p \uparrow (q \uparrow r)) \oplus ((p \uparrow q) \uparrow r)$
- $\Upsilon((p \downarrow \neg(q \uparrow r)) \oplus (\neg(p \downarrow q) \uparrow \neg(p \downarrow r)))$  is  $((p \downarrow (q \uparrow r)) \oplus ((p \downarrow q) \uparrow (p \downarrow r)))$
- $\Upsilon((p \uparrow \neg(q \downarrow r)) \oplus (\neg(p \uparrow q) \downarrow \neg(p \uparrow r)))$  is  $((p \uparrow (q \downarrow r)) \oplus ((p \uparrow q) \downarrow (p \uparrow r)))$

**Lemma 2.14.** If  $\mathcal{A} \perp \mathcal{A}'$  we can construct a rule such that changing  $\mathcal{I}(\mathcal{A}')$  obtain  $\mathcal{A} \parallel \mathcal{B}'$ .

*Proof.* By **Corollary 2.12**  $\mathcal{A} \perp \mathcal{A}'$  can be  $\mathcal{A} \parallel \neg \mathcal{A}'$ , now  $\neg \mathcal{A}'$  is  $\mathcal{B}'$  ■

**Definition 2.15.** If  $\Psi$  is the property mentioned in **Lemma 2.14**, this changes the interpretation of the  $\mathcal{A}'$  final analysis and converts  $\mathcal{A}$  in  $\neg \mathcal{A}'$ . As  $\rho$ -assoc and  $\rho$ -dist properties with NFO have the final analysis with the  $\oplus$  binary operator  $\Psi$  only needs to change the interpretation of  $\oplus$  operator, call this replacement operator  $\uparrow$ .

- $\Psi(((p \downarrow (q \downarrow r)) \oplus ((p \downarrow q) \downarrow r)))$  is  $((p \downarrow (q \downarrow r)) \uparrow ((p \downarrow q) \downarrow r))$
- $\Psi(((p \uparrow (q \uparrow r)) \oplus ((p \uparrow q) \uparrow r)))$  is  $((p \uparrow (q \uparrow r)) \uparrow ((p \uparrow q) \uparrow r))$
- $\Psi(((p \downarrow (q \uparrow r)) \oplus ((p \downarrow q) \uparrow (p \downarrow r))))$  is  $((p \downarrow (q \uparrow r)) \uparrow ((p \downarrow q) \uparrow (p \downarrow r)))$
- $\Psi(((p \uparrow (q \downarrow r)) \oplus ((p \uparrow q) \downarrow (p \uparrow r))))$  is  $((p \uparrow (q \downarrow r)) \uparrow ((p \uparrow q) \downarrow (p \uparrow r)))$

**Theorem 2.16.** NFO  $\uparrow$  perform the same operations of FO  $\leftrightarrow$

*Proof.* By **Definition 1.1**  $\oplus$  is the negation of  $\leftrightarrow$  and how  $\uparrow$  change the interpretation of  $\oplus$  then  $\uparrow$  perform the same operations of  $\leftrightarrow$  ■

Now we can change  $\uparrow$  of NFO for  $\leftrightarrow$  of FO.

**Corollary 2.17.** The next  $\mathcal{A}'$ -wfs are tautologies

- (a)  $\neg(((p \downarrow \neg(q \downarrow r)) \oplus (\neg(p \downarrow q) \downarrow r)))$
- (b)  $\neg(((p \uparrow \neg(q \uparrow r)) \oplus (\neg(p \uparrow q) \uparrow r)))$

- (c)  $\neg(((p \downarrow \neg(q \uparrow r)) \oplus (\neg(p \downarrow q) \uparrow \neg(p \downarrow r))))$   
 (d)  $\neg(((p \uparrow \neg(q \downarrow r)) \oplus (\neg(p \uparrow q) \downarrow \neg(p \uparrow r))))$

In this way we have obtained a  $\rho$ -assoc and  $\rho$ -dist of  $\mathcal{A}$ -wfs  $\mathcal{E} \Sigma$  with NFO in the semantic of the 0-OL with the application of the rules  $\Upsilon$  and  $\Psi$ .

*Note.* The wfs of **Corollary 2.17** encrypted by  $\Upsilon, \Psi$  are.

- (a)  $(p \downarrow (q \downarrow r)) \leftrightarrow ((p \downarrow q) \downarrow r)$   
 (b)  $(p \uparrow (q \uparrow r)) \leftrightarrow ((p \uparrow q) \uparrow r)$   
 (c)  $(p \downarrow (q \uparrow r)) \leftrightarrow ((p \downarrow q) \uparrow (p \downarrow r))$   
 (d)  $(p \uparrow (q \downarrow r)) \leftrightarrow ((p \uparrow q) \downarrow (p \uparrow r))$

*Note.* The rules  $\Upsilon$  and  $\Psi$  are decryptable.

### 3 Sintactical comparison

In [7] is defined  $\mathcal{L}$ -Theory of  $\mathcal{H}\mathcal{A}$  where  $\mathcal{A}$ -f is a formula,  $p, q, \dots$  are symbols

**Definition 3.1** ( $\mathcal{A}$ -f of  $\mathcal{L}_{\mathcal{H}\mathcal{A}}$ ). If  $p$ -f,  $q$ -f of  $\mathcal{L}_{\mathcal{H}\mathcal{A}}$ , the following are formulas of  $\mathcal{L}_{\mathcal{H}\mathcal{A}}$

- $p \downarrow q$  is  $\neg(p \vee q)$
- $p \uparrow q$  is  $\neg(p \wedge q)$
- $p \leftarrow q$  is  $\neg(p \rightarrow q)$
- $p \oplus q$  is  $\neg(p \leftrightarrow q)$

**Proposition 3.2.**  $\mathcal{L}_{\mathcal{H}\mathcal{A}}$  satisfies

- (a)  $\vdash_{\mathcal{L}_{HA}} \neg(((p \downarrow \neg(q \downarrow r)) \oplus (\neg(p \downarrow q) \downarrow r)))$   
 (b)  $\vdash_{\mathcal{L}_{HA}} \neg(((p \uparrow \neg(q \uparrow r)) \oplus (\neg(p \uparrow q) \uparrow r)))$   
 (c)  $\vdash_{\mathcal{L}_{HA}} \neg(((p \downarrow \neg(q \uparrow r)) \oplus (\neg(p \downarrow q) \uparrow \neg(p \downarrow r))))$   
 (d)  $\vdash_{\mathcal{L}_{HA}} \neg(((p \uparrow \neg(q \downarrow r)) \oplus (\neg(p \uparrow q) \downarrow \neg(p \uparrow r))))$

*Proof.* By **Corollary 2.17**  $a, b, c, d$  are tautologies, by completeness theorem [3] [5] [7]  $a, b, c, d$  are theorems of  $\mathcal{L}_{\mathcal{H}\mathcal{A}}$  ■

### 4 Main result

The next formulas are the result of the paper.

- (a)  $\vdash_{\mathcal{L}_{HA}} (p \downarrow \neg(q \downarrow r)) \leftrightarrow (\neg(p \downarrow q) \downarrow r)$   
 (b)  $\vdash_{\mathcal{L}_{HA}} (p \uparrow \neg(q \uparrow r)) \leftrightarrow (\neg(p \uparrow q) \uparrow r)$   
 (c)  $\vdash_{\mathcal{L}_{HA}} (p \downarrow \neg(q \uparrow r)) \leftrightarrow (\neg(p \downarrow q) \uparrow \neg(p \downarrow r))$   
 (d)  $\vdash_{\mathcal{L}_{HA}} (p \uparrow \neg(q \downarrow r)) \leftrightarrow (\neg(p \uparrow q) \downarrow \neg(p \uparrow r))$

## References

- [1] Ben-Ari, M. *Mathematical logic for computer science*, Springer (2012) 16-17
- [2] Bloch, E.D. *Proofs and fundamentals: A first course in abstract mathematics*, Springer-Heidelberg (2011) 19-20
- [3] Dalen, D. *Logic and structure*, Springer-Heidelberg (2011) 46-47
- [4] Hedman, S. *A first course in logic an introduction to model theory, proof theory, computability and complexity*, Oxford University Press (2006) 21
- [5] Kalmár, L. *Über die axiomatisierbarkeit des aussagenkalküls. Acta scientiarum mathematicarum*, Szeged (1935) 222-243
- [6] Manin, Y.I. *A course in mathematical logic for mathematicians*, Springer (2010)
- [7] Mendelson, E. *Introduction to mathematical logic*, Chapman & Hall (1997) 33-42
- [8] Prestel, A. Delzell, C.N. *Mathematical logic and model theory*, Springer (2011)